1.

martedì 3 gennaio 2023 15:09

Sia $w = \frac{(3x^2-y^2)(x^2+y^2)}{x^2 y}dx + \frac{(3y^2-x^2)(x^2+y^2)}{x y^2}dy$

forma differenziale definita in $\Omega = \{(x,y)\in \mathbb{R}^2 \mid x>0, y>0\}$

Stabilire se w è esatta e trovare un potenziale

Sol

$$w = \frac{3x^4+3x^2y^2-x^2y^2-y^4}{x^2y}dx + \frac{3y^2x^2+3y^4-x^4-x^2y^2}{xy^2}dy$$

$$U(x,y) = \int \frac{3x^4+2x^2y^2-y^4}{x^2y}dx = \int \frac{3x^4}{x^2y} + \frac{2x^2y^2}{x^2y} - \frac{y^4}{x^2y}dx =$$

$$= \int \frac{3x^2}{y} + 2y - \frac{y^3}{x^2}dx =$$

$$= \frac{x^3}{y} + 2xy + \frac{y^3}{x} + c(y)$$

$$\Rightarrow U_y = -\frac{x^3}{y^2} + 2x + 3\frac{y^2}{x} + c'(y) = \frac{-x^4+2x^2y^2+3y^4}{xy^2} + c'(y)$$

$$\Rightarrow \frac{3y^4+2x^2y^2-x^4}{xy^2} + c'(y) = \frac{3y^2x^2+3y^4-x^4-x^2y^2}{xy^2}$$

$$\Rightarrow c'(y) = \cancel{3y^2x^2} + \cancel{3y^4} - \cancel{x^4} - \cancel{x^2y^2} - \cancel{3y^4} - \cancel{2x^2y^2} + \cancel{x^4} = 0$$

$$\Rightarrow c(y) = K$$

$$\Rightarrow U(x,y) = \frac{x^3}{y} + 2xy + \frac{y^3}{x} + K$$

É esatta?

É esatta?

$\gamma = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi]$

$$\int_\gamma (-\sin(\theta)) \frac{3\cos^2(\theta) - \sin^2(\theta)}{\cos^2(\theta) \sin(\theta)} + (\cos(\theta)) \frac{3\sin^2(\theta) - \cos^2(\theta)}{\cos(\theta)\sin^2(\theta)} d\theta =$$

$$= \int_\gamma \frac{\sin^2(\theta) - 3\cos^2(\theta)}{\cos^2(\theta)} + \frac{3\sin^2(\theta) - \cos^2(\theta)}{\sin^2(\theta)} d\theta =$$

$$= \int_0^{2\pi} tg^2(\theta) - cotg^2(\theta) \, d\theta = tg(\theta) - \theta + \theta + cotg(\theta) \Big|_0^{2\pi}$$

$$= tg(\theta) + cotg(\theta) \Big|_0^{2\pi} = 0$$

$\Rightarrow$ è esatta

▨

Es 2:

Sia T il dominio nel primo quadrante limitato dalle curve:

$\gamma_1 : \; y = x$

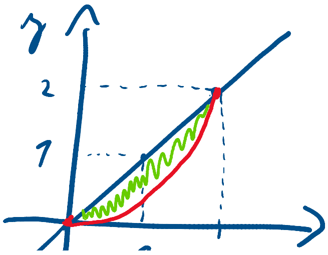
$\gamma_2 : \; x(t) = t^2 + t \; , \; y(t) = t^4 + t \qquad t \in [0,1]$

Calcolare l'area di T

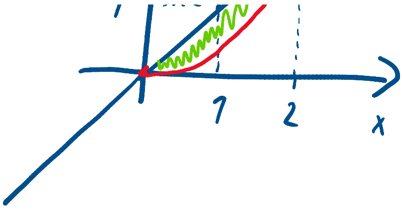
Sol

$\gamma_1 : \; y = x$

$$\gamma_2 : \begin{cases} x = t^2 + t \\ y = t^4 + t \end{cases} \quad t \in [0,1]$$



$\gamma_2 : \{ y = t^4 + t \quad t \in [0,1]$



Da γ_2 vogliamo esprimere la y in funzione di x

Notiamo che $x = t^2 + t$

$$\Rightarrow t^2 + t - x = 0 \Rightarrow t = \frac{-1 \pm \sqrt{1+4x}}{2} = \frac{-1+\sqrt{1+4x}}{2}$$

$$\Rightarrow y = t^4 + t = \left(\frac{-1+\sqrt{1+4x}}{2} \right)^4 + \frac{-1+\sqrt{1+4x}}{2} =$$

$$= \left(\frac{-1+\sqrt{1+4x}}{2} \right)^2 \left(\frac{-1+\sqrt{1+4x}}{2} \right)^2 + \frac{-1+\sqrt{1+4x}}{2} =$$

$$= \left(\frac{1+1+4x-2\sqrt{1+4x}}{4} \right) \left(\frac{1+1+4x-2\sqrt{1+4x}}{4} \right) + \frac{-1+\sqrt{1+4x}}{2} =$$

$$= \left(\frac{2+4x-2\sqrt{1+4x}}{4} \right) \left(\frac{2+4x-2\sqrt{1+4x}}{4} \right) + \frac{-1+\sqrt{1+4x}}{2} =$$

$$= \frac{\overset{\cdot}{4} + \overset{\frown}{8x} - \overset{\#}{4\sqrt{1+4x}} + \overset{\frown}{8x} + 16x^2 - \overset{//}{8x\sqrt{1+4x}} - \overset{\#}{4\sqrt{1+4x}} - \overset{//}{8x\sqrt{1+4x}} + \overset{\cdot}{4} + \overset{\frown}{16x}}{16} + \frac{-1+\sqrt{1+4x}}{2} =$$

$$= \frac{16x^2 + 32x - 16x\sqrt{1+4x} - 8\sqrt{1+4x} + 8}{16} + \frac{-1+\sqrt{1+4x}}{2} =$$

$$= \frac{16x^2 + 32x - 16x\sqrt{1+4x} \cancel{- 8\sqrt{1+4x}} \cancel{+8} \cancel{-8} \cancel{+ 8\sqrt{1+4x}}}{16} =$$

$$= \frac{16x^2 + 32x - 16x\sqrt{1+4x}}{16} = x^2 + 2x - x\sqrt{1+4x}$$

$$\Rightarrow y = x^2 + 2x - x\sqrt{1+4x}$$

$$\Rightarrow y = x^2+2x-x\sqrt{1+4x}$$

$$\text{Area}(T) = \int_0^2 \int_{x^2+2x-x\sqrt{1+4x}}^{x} dy \; dx = \int_0^2 x - x^2 - 2x + x\sqrt{1+4x} \; dx =$$

$$= \int_0^2 -x^2 - x + x\sqrt{1+4x} \; dx =$$

$$= -\frac{x^3}{3} - \frac{x^2}{2} + \frac{2}{12}x(1+4x)^{3/2} - \frac{4}{240}(1+4x)^{5/2} \Big|_0^2 =$$

$$= -\frac{8}{3} - \frac{4}{2} + \frac{4}{12}(9)^{3/2} - \frac{4}{240}(9)^{5/2} + \frac{4}{240} = \frac{3}{10}$$

$$\int \underbrace{x}_{f(x)} \underbrace{\sqrt{1+4x}}_{g'(x) \to g(x) = \frac{2}{12}(1+4x)^{3/2}} = \frac{2}{12}x(1+4x)^{3/2} - \frac{2}{12}\int (1+4x)^{3/2} dx = \frac{2}{12}x(1+4x)^{3/2} - \frac{2}{12}\left(\frac{2}{20}(1+4x)^{5/2}\right) =$$

$$= \frac{2}{12}x(1+4x)^{3/2} - \frac{4}{240}(1+4x)^{5/2}$$

$\blacksquare$

Es 3:

Determinare $f \in C^1(\mathbb{R})$ con $f(0) = 1$ t.c.

$\omega = x f(x) y^2 \, dx - y \log|f(x)| \, dy$ sia esatta in $\mathbb{R}^2$ e si calcoli un potenziale che si annulli in $(0,1)$

SOL

Vogliamo che $\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$

$$\Rightarrow 2x f(x) y = -y \frac{1}{|f(x)|} \cdot \frac{|f(x)|}{f(x)} f'(x) \Rightarrow 2x f(x) y = -y \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad 2x f(x) y = -y \frac{1}{|f(x)|} \cdot \frac{|f(x)|}{f(x)} f'(x) \Rightarrow 2x f(x) y = -y \frac{f'(x)}{f(x)}$$

$$\Rightarrow 2x f^2(x) = -f'(x) \Rightarrow f'(x) = -2x f^2(x)$$

Siamo in presenza di una equazione differenziale di Bernoulli:

$$y' = 0 \cdot y - 2x y^2 \quad \Rightarrow \quad \frac{y'}{y^2} = \frac{0 \cdot y}{y^2} - 2x$$

$$\Rightarrow \frac{y'}{y^2} = 0 \cdot y^{-1} - 2x$$

Sia $z = y^{-1} \Rightarrow z' = -\frac{y'}{y^2}$

$$\Rightarrow -z' = 0 \cdot z - 2x \Rightarrow -z' = -2x \Rightarrow z' = 2x$$

$$\Rightarrow z = \int 2x = x^2 + c$$

$$\Rightarrow z = y^{-1} \Rightarrow y^{-1} = x^2 + c \Rightarrow \frac{1}{y} = x^2 + c \Rightarrow y = \frac{1}{x^2 + c}$$

$$\Rightarrow f(x) = \frac{1}{x^2 + c}$$

$$\begin{cases} f(x) \\ f(0) = 1 \end{cases} \Rightarrow \frac{1}{c} = 1 \Rightarrow c = 1 \Rightarrow f(x) = \frac{1}{x^2+1}$$

$$\Rightarrow \omega = \frac{x y^2}{x^2+1} dx - y \log \left| \frac{1}{x^2+1} \right| dy$$

$$U(x,y) = \int \frac{x y^2}{x^2+1} dx = y^2 \int \frac{x}{x^2+1} dx = \frac{1}{2} y^2 \log(x^2+1) + c(y)$$

$$U_y = y \log(x^2+1) + c'(y)$$

$U_y = y \log(x^2+1) + c'(y)$

Deve essere $U_y = F_2$

$\Rightarrow y \log(x^2+1) + c'(y) = -y \log \left| \frac{1}{x^2+1} \right|$

$\Rightarrow c' = -y \log \left| \frac{1}{x^2+1} \right| - y \log(x^2+1)$

$c' = -y \, (\log (\frac{1}{x^2+1}) + \log(x^2+1) = -y \, (\log(1)) = 0$

$\Rightarrow c(y) = K$

$\Rightarrow U(x,y) = \frac{1}{2} y^2 \log(x^2+1) + K$

$\begin{cases} U(x,y) \\ U(0,1) = 0 \end{cases} \Rightarrow \frac{1}{2} \cdot 1 \cdot \log(0+1) + K = 0 \Rightarrow K = 0$

$\Rightarrow U(x,y) = \frac{1}{2} y^2 \log(x^2+1)$

▨

Es 4:

Sia $\alpha > 0$ e sia T il sottoinsieme di R^3 delimitato dalla semisfera superiore di centro l'origine e raggio 1 escluso il cono $z = \sqrt{x^2+y^2}$

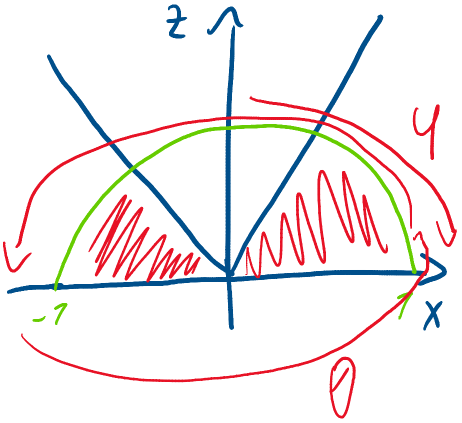
Calcolare $\iiint_T (x^2+y^2+z^2)^\alpha \, dx \, dy \, dz$

SOL

Usiamo le coordinate sferiche

Usiamo le coordinate sferiche

$$\begin{cases} x = \rho \sin(\varphi)\cos(\theta) \\ y = \rho \sin(\varphi)\sin(\theta) \\ z = \rho\cos(\varphi) \end{cases} \qquad \begin{aligned} &\rho \in [0,1] \\ &\varphi \in [\tfrac{\pi}{4}, \tfrac{\pi}{2}] \\ &\theta \in [0, 2\pi] \end{aligned}$$



$$\iiint_T (x^2+y^2+z^2)^{\alpha}\, dx\, dy\, dz$$

$$= \int_0^1 \int_{\pi/2}^{\pi/4} \int_0^{2\pi} \rho^4 \sin(\varphi)\, \partial\theta\, \partial\varphi\, \partial\rho =$$

$$= \int_0^1 \int_{\pi/2}^{\pi/4} 2\pi \rho^4 \sin(\varphi)\, \partial\varphi\, \partial\rho = \int_0^1 -2\pi \rho^4 \cos(\varphi) \Big|_{\pi/2}^{\pi/4} \partial\rho =$$

$$= \int_0^1 -2\pi \frac{\sqrt{2}}{2} \rho^4\, \partial\rho = \int_0^1 -\sqrt{2}\pi \rho^4\, \partial\rho = -\sqrt{2}\pi \frac{\rho^5}{5}\Big|_0^1 = -\sqrt{2}\pi$$

▨

Es 5:

Determinare $f \in C^1(\mathbb{R})$ t.c. $f(0) = 1$ e $\omega = \dfrac{2xy\, f(x)}{1+f^2(x)}\, dx + \operatorname{arctg}(f(x))\, dy$ sia esatta.

Sol

$$\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$$

$$\Rightarrow \quad \frac{2x\, f(x)}{1+f^2(x)} = \frac{1}{1+f^2(x)} \cdot f'(x) \quad \Rightarrow \quad f'(x) = 2x\, f(x)$$

$$\Rightarrow \quad y' = 2x\, y \quad \Rightarrow \quad \frac{\partial y}{\partial x} = 2x\, y \quad \Rightarrow \quad \frac{1}{y}\, \partial y = 2x\, \partial x$$

$$\Rightarrow \quad \int \frac{1}{y}\, \partial y = \int 2x\, \partial x \quad \Rightarrow \quad \log(y) = x^2 + c \quad \Rightarrow \quad y = e^{x^2+c}$$

$$\Rightarrow y = e^{x^2} + c \quad \Rightarrow \quad f(x) = e^{x^2} + c$$

$$\begin{cases} f(x) \\ f(0) = 1 \end{cases} \quad \Rightarrow \quad e^0 + c = 1 \quad \Rightarrow \quad 1 + c = 1 \quad \Rightarrow \quad c = 0$$

$$\Rightarrow \quad f(x) = e^{x^2}$$

▨

Es 6:

Calcolare $\int_{\Sigma} \frac{1}{z^4}\, d\sigma$ dove $\Sigma = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, z = \frac{1}{\sqrt{x^2+y^2}},\ 1 \le z \le 2 \right\}$

Sol

$$\int_{\Sigma} \frac{1}{z^4}\, d\sigma$$

Da Σ sappiamo che $1 \le z \le 2$ $\Rightarrow$

$$\Rightarrow \quad 1 \le \frac{1}{\sqrt{x^2+y^2}} \le 2 \quad \Rightarrow \quad 1 \ge x^2 + y^2 \ge \frac{1}{4}$$

$\cup\, x^2+y^2$ (

$$\Rightarrow \quad \frac{1}{4} \leq x^2+y^2 \leq 1$$

$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta) \\ z = \dfrac{1}{\rho} \end{cases} \qquad \begin{array}{l} \rho \in \left[\frac{1}{2}, 1\right] \\ \theta \in [0, 2\pi] \end{array}$$

Dobbiamo determinare lo Jacobiano:

$$\varphi_\rho = \left(\cos(\theta), \sin(\theta), -\frac{1}{\rho^2}\right)$$

$$\varphi_\theta = \left(-\rho\sin(\theta), \rho\cos(\theta), 0\right)$$

$$\varphi_\rho \wedge \varphi_\theta = \begin{vmatrix} i & j & k \\ \cos(\theta) & \sin(\theta) & -\frac{1}{\rho^2} \\ -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{vmatrix} = \left(\frac{1}{\rho}\cos(\theta), -\frac{\sin(\theta)}{\rho}, \rho\right)$$

$$\Rightarrow \|\varphi_\rho \wedge \varphi_\theta\| = \sqrt{\frac{1}{\rho^2}+\rho^2} = \frac{1}{\rho}\sqrt{1+\rho^4}$$

Allora

$$\int_\Sigma \frac{1}{z^4}\, d\sigma = \int_\Omega \rho^4 \cdot \frac{1}{\rho}\sqrt{1+\rho^4} = \int_\Omega \rho^3\sqrt{1+\rho^4}$$

$$= \int_{1/2}^{1}\int_0^{2\pi} \rho^3\sqrt{1+\rho^4}\, d\theta\, d\rho = 2\pi\int_{1/2}^{1} \rho^3\sqrt{1+\rho^4}\, d\rho$$

$$= 2\pi\left[\frac{1}{6}(1+\rho^4)^{3/2}\right]\Big|_{1/2}^{1}$$

$$= 2\pi \left(\frac{1}{6} (2)^{3/2} - \frac{1}{6} \left(1 + \frac{1}{16}\right)^{3/2} \right) = \frac{\pi}{3} \left(2\sqrt{2} - \frac{17\sqrt{17}}{64} \right)$$

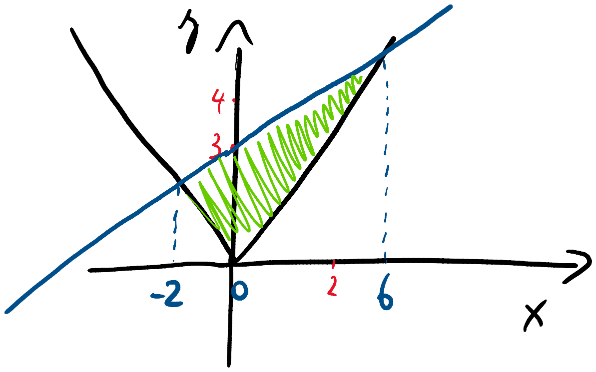
▨

Es 3 :

Calcolare $\int_{\Omega} x|z| \, dx\, dy\, dz$ dove $\Omega = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \sqrt{x^2+z^2} < y < \frac{1}{2}x + 3 \right\}$

Sol

Se $z=0 \Rightarrow |x| < y < \frac{1}{2}x+3$



$$\begin{cases} y = |x| \\ y = \frac{1}{2}x+3 \end{cases} \Rightarrow x = \frac{1}{2}x+3 \to \frac{1}{2}x = 3 \to x = 6$$

$$-x = \frac{1}{2}x+3 \to -\frac{3}{2}x = 3 \to \frac{3}{2}x = -3 \to x = -2$$

$$\Rightarrow x \in [-2, 6]$$

Sappiamo che $\sqrt{x^2+z^2} < y < \frac{1}{2}x+3$

$$\Rightarrow \int_{\Omega} x|z| = \iint_{A} \int_{\sqrt{x^2+z^2}}^{\frac{1}{2}x+3} x|z| \, dy \, dx\, dz =$$

$$= \int_A x|z|\, y \Big|_{\sqrt{x^2+z^2}}^{\frac{1}{2}x+3} dx\, dz = \int_A x|z| \left(\frac{1}{2}x+3-\sqrt{x^2+z^2}\right) dx\, dz =$$

$$= \int_A \frac{1}{2}x^2|z| + 3x|z| - x|z|\sqrt{x^2+z^2}\; dx\, dz =$$

$$= 2\int_A \frac{x^2}{2} z + 3xz - xz\sqrt{x^2+z^2}\; dx\, dz$$

Da Ω sappiamo che $\sqrt{x^2+z^2} \leq \frac{1}{2}x+3$

$$\Rightarrow \quad 0 \leq \sqrt{x^2+z^2} \leq \frac{1}{2}x+3 \quad \Rightarrow \quad 0 \leq x^2+z^2 \leq \frac{1}{4}x^2+9+3x$$

$$\Rightarrow \quad 0 \leq z^2 \leq -\frac{3}{4}x^2+9+3x \quad \Rightarrow \quad 0 \leq z \leq \sqrt{-\frac{3}{4}x^2+3x+9}$$

A questo punto:

$$2\int_A \frac{x^2}{2}z + 3xz - xz\sqrt{x^2+z^2}\; dx\, dz =$$

$$= 2\int_{-2}^{6} \int_{0}^{\sqrt{-\frac{3}{4}x^2+3x+9}} \frac{x^2}{2}z + 3xz - xz\sqrt{x^2+z^2}\; dz\, dx =$$

$$= 2\int_{-2}^{6} \frac{x^2}{2}\frac{z^2}{2} + 3x\frac{z^2}{2} - \frac{1}{3}x\left(x^2+z^2\right)^{3/2} \Bigg|_{0}^{\sqrt{-\frac{3}{4}x^2+3x+9}} dx =$$

$$= 2\int^{6} x^2 \; -\frac{3}{4}x^2+3x+9 \; + 3x \; -\frac{3}{4}x^2+3x+9 \; - \frac{1}{}x\left(x^2+\left(-\frac{3}{}x^2+3x+9\right)\right)^{3/2} + \frac{1}{}x^4 \; dx$$

$$= 2\int_{-2}^{6} \frac{x^2}{2}\,\frac{-\frac{3}{4}x^2+3x+9}{2} + 3x\,\frac{-\frac{3}{4}x^2+3x+9}{2} - \frac{1}{3}x\left(x^2+\left(-\frac{3}{4}x^2+3x+9\right)\right)^{3/2} + \frac{1}{3}x^4\,dx$$

$$= 2\int_{-2}^{6} \frac{-\frac{3}{4}x^4+3x^3+9x^2}{4} + \frac{-\frac{9}{4}x^3+9x^2+27x}{2} - \frac{1}{3}x\left(x^2-\frac{3}{4}x^2+3x+9\right)^{3/2} + \frac{1}{3}x^4\,dx =$$

$$= 2\int_{-2}^{6} -\frac{3}{16}x^4+\frac{3}{4}x^3+\frac{9}{4}x^2-\frac{9}{8}x^3+\frac{9}{2}x^2+\frac{27}{2}x-\frac{1}{3}x\left(\frac{1}{2}x+3\right)^3+\frac{1}{3}x^4\,dx =$$

$$= 2\int_{-2}^{6} -\frac{3}{16}x^4+\frac{3}{4}x^3+\frac{9}{4}x^2-\frac{9}{8}x^3+\frac{9}{2}x^2+\frac{27}{2}x-\frac{1}{3}x\left(\frac{1}{8}x^3+27+\frac{9}{4}x^2+\frac{27}{2}x\right)+\frac{1}{3}x^4\,dx =$$

$$= 2\int_{-2}^{6} \frac{5}{48}x^4-\frac{9}{8}x^3+\frac{9}{4}x^2+\frac{9}{2}x\,dx =$$

$$= 2\left[\frac{5}{48}\cdot\frac{x^5}{5}-\frac{9}{8}\frac{x^4}{4}+\frac{9}{4}\frac{x^3}{3}+\frac{9}{2}\frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{256}{3}$$

▨

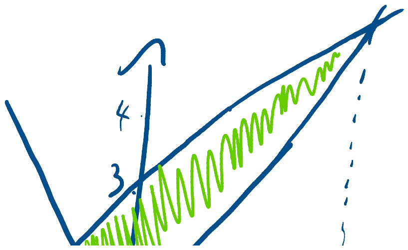
Esercizio:

Calcolare

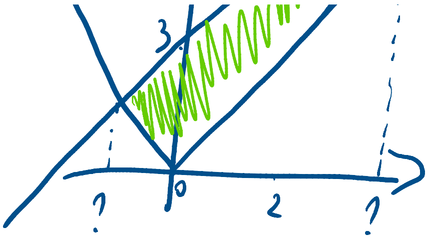
$$\int_{\Omega} x|z|\,dx\,dy\,dz \qquad \text{dove } \Omega = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \sqrt{x^2+z^2} < y < \frac{1}{2}x+3\right\}$$

Sol

Se $z=0 \Rightarrow |x| < y < \frac{1}{2}x+3$



Se $z<0 \quad \Rightarrow \quad |x| < y < \frac{1}{2}x+3$



$$\begin{cases} -x = \frac{1}{2}x+3 \\ x = \frac{1}{2}x+3 \end{cases} \Rightarrow \begin{cases} \frac{1}{2}x+x+3=0 \\ \frac{1}{2}x-x+3=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{3}{2}x+3=0 \\ -\frac{1}{2}x+3=0 \end{cases} \Rightarrow \begin{cases} \frac{3}{2}x=-3 \Rightarrow x=-3\cdot\frac{2}{3}=-2 \\ \frac{1}{2}x=3 \Rightarrow x=6 \end{cases}$$

Se $z>0$:

$$\sqrt{x^2+z^2} < y \Rightarrow 0 < z < \sqrt{y^2-x^2}$$

$$\Rightarrow \int_\Omega x\,|z|\,dx\,dy\,dz = 2\int_\Omega x\,z\,dx\,dy\,dz$$

Scomponiamo Ω in Σ_1 e Σ_2:

$$\Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x\in[-2,0],\ -x<y<\frac{1}{2}x+3,\ 0<z<\sqrt{y^2-x^2} \right\}$$

$$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x\in[0,6],\ x<y<\frac{1}{2}x+3,\ 0<z<\sqrt{y^2-x^2} \right\}$$

$$\Rightarrow \int_{-2}^{0}\int_{-x}^{\frac{1}{2}x+3}\int_{0}^{\sqrt{y^2-x^2}} x\,z\,dz\,dy\,dx = \int_{-2}^{0}\int_{-x}^{\frac{1}{2}x+3} x\,\frac{z^2}{2}\Big|_{0}^{\sqrt{y^2-x^2}}\,dy\,dx =$$

$-2 \quad -x$

$$= \int_{-2}^{0} \int_{-x}^{\frac{1}{2}x+3} x\left(\frac{y^2-x^2}{2}\right) dy\, dx = \int_{-2}^{0} \int_{-x}^{\frac{1}{2}x+3} \frac{x}{2}y^2 - \frac{x^3}{2}\, dy\, dx =$$

$= \dots$ *da finire*

▨

Esercizio :

Sia Γ il dominio definito da $D = \left\{ \binom{x}{y} \in \mathbb{R}^2 \,\middle|\, x^2 \leq 3,\ \frac{1}{\sqrt{3}}|x| \leq y \leq \sqrt{4-x^2} \right\}$

Calcolare $\displaystyle\int_{\partial D^+} (\arctan(3x) - 5y)\, dx + (2x + \log(y))\, dy$

SOL

Usiamo Gauss-Green "al rovescio"

$$\Rightarrow \int_D \left[\frac{\partial}{\partial x}(2x + \log(y)) - \frac{\partial}{\partial y}(\arctan(3x) - 5y)\right] dx\, dy =$$

$$= \int_D 2 - (-5)\, dx\, dy = 7 \int_D dx\, dy$$

Ora $D = \left\{ \binom{x}{y} \in \mathbb{R}^2 \,\middle|\, x^2 \leq 3,\ \frac{1}{\sqrt{3}}|x| \leq y \leq \sqrt{4-x^2} \right\}$

$\Rightarrow \quad -\sqrt{3} \leq x \leq \sqrt{3}$

$(y \geq \frac{1}{\sqrt{3}} x$

$\Rightarrow \quad -\sqrt{3} \leq x \leq \sqrt{3}$

$y \geq \frac{1}{\sqrt{3}}|x| \Rightarrow \begin{cases} y \geq \frac{1}{\sqrt{3}}x \\ -y \leq \frac{1}{\sqrt{3}}x \Rightarrow y \geq -\frac{1}{\sqrt{3}}x \end{cases}$

$y \leq \sqrt{4-x^2} \Rightarrow y^2 \leq 4-x^2 \Rightarrow y^2+x^2 \leq 4 \Rightarrow x^2+y^2 \leq 4$

$D = D_1 \cup D_2$

dove $D_1 = \{ x \in [-\sqrt{3}, 0] , \quad -\frac{1}{\sqrt{3}}x \leq y \leq \sqrt{4-x^2} \}$

$D_2 = \{ x \in [0, \sqrt{3}] \quad , \quad \frac{1}{\sqrt{3}}x \leq y \leq \sqrt{4-x^2} \}$

$\Rightarrow D_1 = \{ x \in [-\sqrt{3}, 0] , \; y \geq -\frac{1}{\sqrt{3}}x , \quad x^2+y^2 \leq 4 \}$

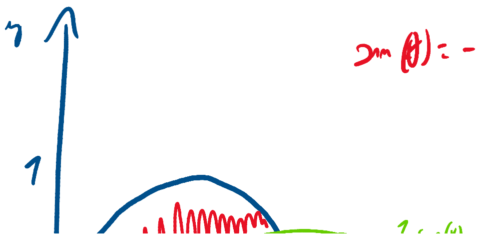
$D_2 = \{ x \in [0, \sqrt{3}] , \; y \geq \frac{1}{\sqrt{3}}x \quad , \; x^2+y^2 \leq 4 \}$

$\Rightarrow 1 \int_D dx\,dy = 1 \int_{D_1} dx\,dy + 1 \int_{D_2} dx\,dy$

$\cdot \int_{D_1} dx\,dy$

in polari: $\begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta) \end{cases} \Rightarrow \begin{array}{l} -\sqrt{3} \leq \rho\cos(\theta) \leq 0 \\ \rho\sin(\theta) \geq -\frac{1}{\sqrt{3}}\rho\cos(\theta) \end{array}$

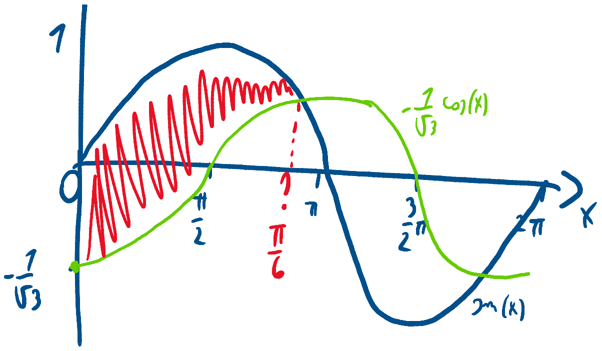
$\Rightarrow \begin{cases} \frac{-\sqrt{3}}{\cos(\theta)} \leq \rho \leq 0 \quad \Rightarrow \rho \in \left[-\frac{\sqrt{3}}{\cos(\theta)} , 0\right] \\ \sin(\theta) \geq -\frac{1}{\sqrt{3}}\cos(\theta) \quad \Rightarrow \theta \in \left[0, \frac{\pi}{6}\right] \end{cases}$



$\sin(\theta) = -\frac{1}{\sqrt{3}}\cos(\theta) \Rightarrow -\tan(\theta) = \frac{1}{\sqrt{3}}$

$\tan(\theta) = -\frac{1}{\sqrt{3}} \quad \Rightarrow \theta = -\frac{\pi}{6}$

$\Rightarrow \pi$



$\sqrt{3} \Rightarrow U - - 6$

$\Rightarrow \frac{\pi}{6}$

$$\Rightarrow \int_{D_1} dx\,dy = \int_0^{\pi/6}\int_{-\frac{\sqrt{3}}{\cos(\theta)}}^{0} \rho \, d\rho \, d\theta = \int_0^{\pi/6} \frac{\rho^2}{2}\Big|_{-\frac{\sqrt{3}}{\cos(\theta)}}^{0} d\theta =$$

$$= \int_0^{\frac{\pi}{6}} - \frac{\frac{3}{\cos^2(\theta)}}{2} = \int_0^{\pi/6} - \frac{3}{2\cos^2(\theta)} = -\frac{3}{2}\int_0^{\pi/6} \underbrace{\frac{1}{\cos^2(\theta)}}_{\operatorname{sen}(\theta)\cdot(\cos(\theta))^{-1}}$$

$$\left[(\cos(\theta))^{-1} \to -(\cos(\theta))^{-2}\cdot(-\operatorname{sen}(\theta)) = \operatorname{sen}(\theta)\cdot\frac{1}{\cos^2(\theta)} = \frac{\operatorname{sen}(\theta)}{\cos^2(\theta)}\right]$$

$$\operatorname{sen}(\theta)\cdot(\cos(\theta))^{-1} \to \cos(\theta)\cdot\cos(\theta)^{-1} + \operatorname{sen}(\theta)\cdot\frac{\operatorname{sen}(\theta)}{\cos^2(\theta)} =$$

$$= 1 + \frac{\operatorname{sen}^2(\theta)}{\cos^2(\theta)} = \frac{\cos^2(\theta)+\operatorname{sen}^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$= -\frac{3}{2}\left(\operatorname{sen}(\theta)\cdot\frac{1}{\cos(\theta)}\right)\Big|_0^{\pi/6} = -\frac{3}{2}\left(\frac{\sqrt{3}}{3}\right) = -\frac{\sqrt{3}}{2}\cdot 1 = -\frac{1}{2}\sqrt{3}$$

$$\cdot \int_{D_2} dx\,dy =$$

in polari: $\begin{cases} x = \rho\cos(\theta) \\ y = \rho\operatorname{sen}(\theta) \end{cases}$

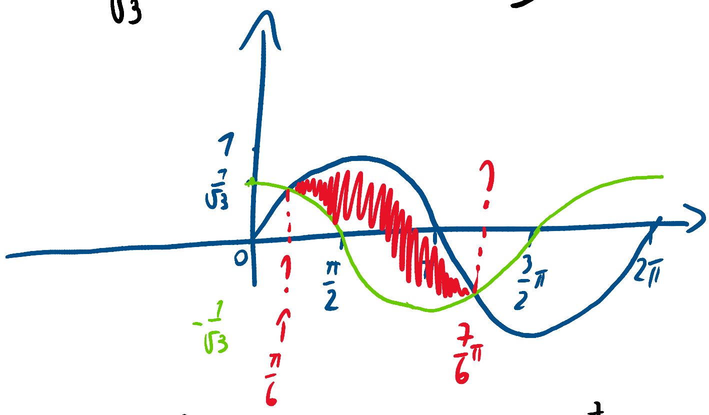
$0 \le \rho\cos(\theta) \le \sqrt{3}$

$\rho\operatorname{sen}(\theta) \ge \frac{1}{\sqrt{3}}\rho\cos(\theta)$

$$\Rightarrow 0 \le \rho \le \frac{\sqrt{3}}{\cos(\theta)}$$

$\operatorname{sen}(\theta) \ge \frac{1}{\sqrt{3}}\cos(\theta) \Rightarrow \theta \in \left[\frac{\pi}{6}, \frac{7}{6}\pi\right]$

$\sin(\theta) \geq \frac{1}{\sqrt{3}} \cos(\theta) \quad \Rightarrow \quad \theta \in \left[\frac{\pi}{6}, \frac{7}{6}\pi\right]$



$$\Rightarrow \int_{D_2} dx\,dy = \int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \int_0^{\frac{\sqrt3}{\cos(\theta)}} \rho \, d\rho \, d\theta = \int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \frac{\rho^2}{2} \Bigg|_0^{\sqrt3/\cos(\theta)} =$$

$$= \int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \frac{3}{2\cos^2(\theta)} \, d\theta = \frac{3}{2} \int_{\frac{\pi}{6}}^{7/6\pi} \frac{1}{\cos^2(\theta)} \, d\theta =$$

$$= \frac{3}{2} \left(\sin(\theta) \cdot \frac{1}{\cos(\theta)} \right) \Bigg|_{\frac{\pi}{6}}^{7/6\pi} = \frac{3}{2}\left(\frac{\sqrt3}{3} - \frac{\sqrt3}{3}\right) = 0 \quad \cdot 7 = 0$$

$$\Rightarrow 7 \int_D dx\,dy = -\frac{7}{2}\sqrt3$$

▨

Es 0.5 :

Calcolare il flusso di $F(x,y,z) = (z^2 \sin(y), 2x^2 z^3, z\sqrt{x^2+y^2})$ uscente dalla superficie del cilindro:

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+y^2 \leq 2x, \; z \in [0,7] \right\}$$

SOL

Per il TEO Della Divergenza

$$\Rightarrow \int_S \text{div}(F) = \int_S \sqrt{x^2+y^2}\, dx\, dy\, dz$$

Parametrizzo in coordinate cilindriche:

$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho \sin(\theta) \\ z = z \end{cases} \qquad \begin{matrix} \rho \in [0,1] \\ \theta \in [0,2\pi] \\ z \in [0,1] \end{matrix}$$

$$\Rightarrow \int_S \sqrt{x^2+y^2} = \int_0^1 \int_0^{2\pi} \int_0^1 \rho^2 \, d\rho \, d\theta \, dz =$$

$$= \int_0^1 \int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^1 \, d\theta\, dz = \int_0^1 \int_0^{2\pi} \frac{1}{3} \, d\theta \, dz = \frac{2}{3}\pi$$

▨

Es 0.5 (Metodo alternativo):

$F = (z^2 \sin(y),\ 2x^2 z^3,\ z\sqrt{x^2+y^2})$

$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+y^2 \leq 2x,\ z \in [0,1] \right\}$

SOL

Parametrizzo S in tre superfici $\Sigma_1, \Sigma_2, \Sigma_3$:

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = 0 \end{cases}, \quad \varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = 1 \end{cases}, \quad \Gamma(\theta, z) = \begin{cases} x = 1 + \cos(\theta) \\ y = \sin(\theta) \\ z = z \end{cases}$$

$\psi_x = (1,0,0)$ $\qquad$ $\varphi_x = (1,0,0)$ $\qquad$ $\Gamma_\theta = (-\sin(\theta), \cos(\theta), 0)$

$\psi_y = (0,1,0)$ $\qquad$ $\varphi_y = (0,1,0)$ $\qquad$ $\Gamma_z = (0, 0, 1)$

$\psi_x \wedge \psi_y = (0,0,1)$ $\qquad$ $\varphi_x \wedge \varphi_y = (0,0,1)$ $\qquad$ $\Gamma_\theta \wedge \Gamma_z = (\cos(\theta), \sin(\theta), 0)$

$N_1 = (0,0,-1)$ $\qquad$ $N_2 = (0,0,1)$

$$\int_{\Sigma_1} F\cdot n = \int_{\Sigma_1} F(\psi)\cdot N_1 = \int_{\Sigma_1} \begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\-1\end{pmatrix} = 0$$

$$\int_{\Sigma_2} F\cdot n = \int_{\Sigma_2} F(\psi)\cdot N_2 = \int_{\Sigma_2} \begin{pmatrix}\sin(y)\\2x^2\\\sqrt{x^2+y^2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \int_{\Sigma_2}\sqrt{x^2+y^2}$$

$$\int_{\Sigma_3} F\cdot n = \int_{\Sigma_3} F(\Gamma)\cdot N_3 = \int_{\Sigma_3} \begin{pmatrix} z^2\sin(\sin(\theta))\\ 2z^3(1+\cos(\theta))^2\\ z\sqrt{(1+\cos(\theta))^2+\sin^2(\theta)}\end{pmatrix}\begin{pmatrix}\cos(\theta)\\\sin(\theta)\\0\end{pmatrix} =$$

$$= \int_{\Sigma_3} z^2\cos(\theta)\sin(\sin(\theta)) + 2z^3\sin(\theta)\left(1+\cos^2(\theta)+2\cos(\theta)\right)$$

$$= \int_{\Sigma_3} z^2\cos(\theta)\sin(\sin(\theta)) + 2z^3\sin(\theta) + 2z^3\cos^2(\theta)\sin(\theta) + 4z^3\cos(\theta)\sin(\theta)$$

$$= \int_0^1\int_0^{2\pi} z^2\cos(\theta)\sin(\sin(\theta)) + 2z^3\sin(\theta) + 2z^3\cos^2(\theta)\sin(\theta) + 4z^3\cos(\theta)\sin(\theta)\, d\theta\, dz =$$

$$= \int_0^1\left(\int_0^{2\pi} z^2\cos(\theta)\sin(\sin(\theta)) + \cancel{\underbrace{\int_0^{2\pi} 2z^3\sin(\theta)}_{\begin{array}{c}\parallel\\ -2z^3\cos(\theta)\end{array}}} + \cancel{\underbrace{\int_0^{2\pi} 2z^3\cos^2(\theta)\sin(\theta)}_{\begin{array}{c}\parallel\\ -\frac{2z^3\cos^3(\theta)}{3}\end{array}}} + \cancel{\underbrace{\int_0^{2\pi} 4z^3\cos(\theta)\sin(\theta)}_{\begin{array}{c}\parallel\\ \frac{4z^3\sin^2(\theta)}{2}\end{array}}}\right) dz$$

$$= \int_0^1\left(\int_0^{2\pi} z^2\cos(\theta)\sin(\sin(\theta))\, d\theta\right) =$$

= Posto $t = \sin(\theta)$

$\Rightarrow dt = \cos(\theta)\, d\theta$

$$\Rightarrow \int_0^1\int_0^0 z^2\sin(t)\, dt = 0$$

- $\int_{\Sigma_2}\sqrt{x^2+y^2}$

in polari: $\left\{ x = 1+\rho\cos(\theta) \qquad \rho\in[0,1]\right.$

in polari:

$$\begin{cases} x = 1 + \rho\cos(\theta) & \rho \in [0,1] \\ y = \rho\sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \int_{\Sigma_2} \sqrt{x^2+y^2} = \int_{\Sigma_2} \rho\sqrt{1+2\rho\cos(\theta)+\rho^2}\ \partial\rho\,\partial\theta = \int_0^{2\pi}\int_0^1 \rho\sqrt{1+2\rho\cos(\theta)+\rho^2}\ \partial\rho\,\partial\theta$$

$= ???$

▨

2.

giovedì 5 gennaio 2023 15:02

Esercizio :

Si consideri il campo vettoriale $F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ dato da:

$$F(x,y,z) = \left(X(x,y,z),\ x^2+2yz,\ y^2-z^2\right)$$

(i) Dimostrare che esistono funzioni $X \in C^1$ che rendono $\vec{F}$ conservativo, trovare tutte le possibili funzioni

(ii) Dimostrare che $\exists!\, X$ come nel punto (i) che per di più è nulla sull'asse x, determinarla

(iii) Determinare i potenziali di F con X come nel punto (ii)

Sol

(i):
$$\mathrm{rot}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ X & x^2+2yz & y^2-z^2 \end{vmatrix} = (2y-2y,\ 0+X_z,\ 2x-X_y)$$

$$= (0,\ X_z,\ 2x-X_y)$$

Voglio che $\mathrm{rot}(F)=0$

$$\Rightarrow \begin{pmatrix} 0 \\ X_z \\ 2x-X_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0=0 \\ X_z = 0 \\ 2x-X_y=0 \end{cases} \Rightarrow \begin{cases} X = \Phi(x,y) \\ X_y = 2x \end{cases}$$

dalla 3ª $\Rightarrow X = \int 2x \, dy = 2xy + \alpha(x)$

da 2ª $\Rightarrow \Phi(x,y) = 2xy + \alpha(x)$

$$\Rightarrow F(x,y,z) = \left(2xy+\alpha(x),\ x^2+2yz,\ y^2-z^2\right)$$

(ii): Vogliamo $X(x,0,0) \equiv 0$

$$\Rightarrow \quad 2x\overset{0}{\cancel{y}} + \alpha(x) \equiv 0 \quad \Rightarrow \quad \alpha(x) = 0$$

$$\Rightarrow \quad X = 2xy$$

(iii): Sia $F(x,y,z) = (2xy, x^2+2yz, y^2-z^2)$

$$U(x,y) = \int 2xy\,dx = x^2y + \alpha(y,z)$$

- $U_y = x^2 + \alpha_y$

Sia $U_y = x^2+2yz \quad \Rightarrow \quad x^2+\alpha_y = x^2+2yz$

$$\Rightarrow \quad \alpha_y = 2yz \quad \Rightarrow \quad \alpha = \int 2yz\,dy = y^2z + \beta(z)$$

$$\Rightarrow \quad \alpha_z = y^2 + \beta'(z)$$

Voglio $\alpha_z = y^2-z^2 \quad \Rightarrow \quad y^2+\beta'(z) = y^2-z^2$

$$\Rightarrow \quad \beta'(z) = -z^2 \quad \Rightarrow \quad \beta(z) = \int -z^2\,dz = -\frac{z^3}{3} + c$$

$$\Rightarrow$$

$$U(x,y,z) = x^2y + y^2z - \frac{z^3}{3} + c$$

▨

Esercizio 7 :

Verificare che ciascuna delle seguenti forme differenziali in $\mathbb{R}^2$ è esatta e determinare una primitiva

(a) $\omega(x,y) = \sin(x)\,dx + \cos(y)\,dy$

(b) $\omega(x,y) = [x^2y + y^2 + 7]\,dx + [\frac{x^3}{3} + 2xy]\,dy$

(c) $\omega(x,y) = (2e^y - ye^x)\,dx + (2xe^y - e^x)\,dy$

(d) $\omega(x,y) = \frac{1}{1+y^2}\,dx - \frac{2xy}{(1+y^2)^2}\,dy$

Sol

(a): Deve essere $\frac{\partial}{\partial y}F_1 = \frac{\partial}{\partial x}F_2$

$\Rightarrow \quad 0 = 0$

Troviamo una primitiva

$U(x,y) = \int F_1\,dx = \int \sin(x)\,dx = -\cos(x) + c(y)$

$U_y = c'(y)$

Deve risultare che $U_y = F_2$

$\Rightarrow \quad c'(y) = \cos(y) \quad \Rightarrow \quad c(y) = \int \cos(y)\,dy$

$\Rightarrow \quad c(y) = \sin(y) + K$

Allora

$U(x,y) = -\cos(x) + \sin(y) + K$

Per le altre si procede in modo analogo

▨

Es 2 :

Determinare α t.c. $\omega = \frac{x-y}{(x^2+y^2)^\alpha} dx + \frac{x+y}{(x^2+y^2)^\alpha} dy$

sia chiusa in $\mathbb{R}^2 \setminus \{(0,0)\}$

Per tali valori di α verificarne l'esattezza

Sol

Vogliamo che $\frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} F_2$

$$\Rightarrow \frac{\partial}{\partial y} F_1 = \frac{-(x^2+y^2)^\alpha - (x-y)\,\alpha\,(x^2+y^2)^{\alpha-1} \cdot 2y}{(x^2+y^2)^{2\alpha}} = \frac{-x^2-y^2-2\alpha y(x-y)}{(x^2+y^2)^{2\alpha}} \cdot (x^2+y^2)^{\alpha-1}$$

$$\frac{\partial}{\partial x} F_2 = \frac{(x^2+y^2)^\alpha - (x+y)\,\alpha\,(x^2+y^2)^{\alpha-1} \cdot 2x}{(x^2+y^2)^{\alpha}} = \frac{x^2+y^2-2\alpha x(x+y)}{(x^2+y^2)^{2\alpha}} \cdot (x^2+y^2)^{\alpha-1}$$

$$\Rightarrow -x^2-y^2-2\alpha y(x-y) = x^2+y^2-2\alpha x(x+y)$$

$$\Rightarrow 2x^2+2y^2+2\alpha y(x-y) - 2\alpha x(x+y) = 0$$

$$\Rightarrow \dots \Rightarrow 2x^2(1-\alpha) + 2y^2(1-\alpha) = 0$$

$\Rightarrow (1-\alpha)(2x^2+2y^2)=0 \quad \Rightarrow \quad \alpha=1$

ω è chiusa $\iff \alpha=1$

Se $\alpha \neq 1 \Rightarrow \omega$ non è esatta

Se $\alpha=1 \Rightarrow \omega = \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy$

Ci chiediamo se $\int_{\bar{\gamma}} \omega = 0$

Sia $\gamma = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$

$$\Rightarrow \int_{\bar{\gamma}} \omega = \int_0^{2\pi} F_1(x(t),y(t))\, x'(t) + F_2(x(t),y(t))\, y'(t)\, dt =$$

$$= \int_0^{2\pi} (\cos(t)-\sin(t))(-\sin(t)) + (\cos(t)+\sin(t))\cos(t)\, dt =$$

$$= \ldots = \int_0^{2\pi} dt = 2\pi \neq 0 \quad \Rightarrow \omega \text{ non è esatta}$$

▨

Esercizio 3 :

Sia $F(x,y) = (P(x,y), Q(x,y))$

dove $P(x,y) = \frac{(3x^2-y^2)(x^2+y^2)}{x^2y}$, $Q(x,y) = \frac{(3y^2-x^2)(x^2+y^2)}{xy^2}$

$\Rightarrow \omega = P(x,y)\,dx + Q(x,y)\,dy$

in $A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x > 0,\ y > 0 \right\}$

(i) Mostrare che il campo è CONSERVATIVO e trovare i potenziali

(ii) $\int_{\gamma^+} F \cdot d\vec{r}$ dove $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$\gamma(t) = \left(1 + \cos^2(t),\ 1 + \sin^2(t)\right)$

SOL

(i): $P(x,y) = \left(\frac{3}{y} - \frac{y}{x^2}\right)(x^2 + y^2)$

$Q(x,y) = \left(\frac{3}{x} - \frac{x}{y^2}\right)(x^2 + y^2)$

Per essere conservativo deve essere $\frac{\partial}{\partial y} P = \frac{\partial}{\partial x} Q$

* la soluzione ma è così *

Calcoliamo il potenziale:

$$U(x,y) = \int P\,dx = \int \left(\frac{3}{y} - \frac{y}{x^2}\right)(x^2 + y^2)\,dx =$$

$$= \int \frac{3}{y}x^2 - y + 3y - \frac{y^3}{x^2}\,dx =$$

$$= \frac{x^3}{y} + 2xy + \frac{y^3}{x} + c(y)$$

$$\Rightarrow U_y = -\frac{x^3}{y^2} + 2x + \frac{3y^2}{x} + c'(y)$$

Deve essere $U_y = Q(x,y)$

$$\cancel{-\frac{x^3}{y^2}} \cancel{+2x} + \cancel{\frac{3y^2}{x}} + c'(y) = \cancel{3x} + \cancel{\frac{3}{x}y^2} \cancel{-\frac{x^3}{y^2}} \cancel{-x}$$

$$\Rightarrow c'(y) = 0 \quad \Rightarrow c(y) = K$$

$$\Rightarrow U(x,y) = \frac{x^3}{y} + 2xy + \frac{y^3}{x} + K$$

(ii): $\int_\gamma F \cdot d\vec{r}$ con $\gamma(t) = (t + \cos^2(t),\ 1 + \sin^2(t))$, $t \in [0, 2\pi]$

Notiamo che $\gamma(0) = (1,1)$ e $\gamma(2\pi) = (2\pi + 1, 1)$

$\Rightarrow \gamma(0) \neq \gamma(2\pi) \Rightarrow \gamma$ non è chiusa

Inoltre $d\vec{r} = \frac{\gamma'(t)}{\|\gamma'(t)\|} dt$

$\gamma'(t) = (1 - 2\sin(t)\cos(t),\ 2\sin(t)\cos(t))$

$$\int_\gamma F \cdot d\vec{r} = \int_0^{2\pi} \langle F, \gamma'(t) \rangle dt = \int_0^{2\pi} \langle (P,Q), (1 - 2\sin(t)\cos(t), 2\sin(t)\cos(t)) \rangle$$

$= \dots$

OPPURE

$$\int_{\gamma} F \cdot d\vec{r} = U(\gamma(2\pi)) - U(\gamma(0)) = U(2\pi+1, 1) - U(1,1)$$

$$= (2\pi+1)^3 + 2(2\pi+1) + \frac{1}{2\pi+1} - 4$$

▨

3.

venerdì 6 gennaio 2023 14:13

Esercizio 1:

Dimostrare che $w(x,y,z) = (e^x \cos(y) + yz)\,dx + (xz - e^x \sin(y))\,dy + xy\,dz$

è esatta e determinarne una primitiva

SOL

× Copia ×

▨

Esercizio 2:

* Es 1 dell'esercitazione precedente

▨

Esercizio 3:

Trovare a, b t.c. il campo $F: \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}^2$

con $F(x,y) = \left(\dfrac{y^2 + 2xy + ax}{(x^2+y^2)^2} , - \dfrac{x^2 + 2xy + by^2}{(x^2+y^2)^2} \right)$

sia conservativo in $\mathbb{R}^2 \setminus \{(0,0)\}$

Trovare i potenziali del campo

SOL

Vogliamo che $\dfrac{\partial}{\partial y} F_1 = \dfrac{\partial}{\partial x} F_2$

$\Rightarrow$

$$\frac{\partial}{\partial y} F_1 = \frac{(2y+2x)(x^2+y^2)^2 - (y^2+2xy+ax) \cdot 2(x^2+y^2)\,2y}{(x^2+y^2)^4} =$$

$$= \frac{(2y+2x)(x^2+y^2)^2 - 4y(y^2+2xy+ax)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial}{\partial x} F_2 = - \frac{(2x+2y)(x^2+y^2)^2 - (x^2+2xy+by^2)\, 2\, (x^2+y^2) \cdot 2x}{(x^2+y^2)^4} =$$

$$= - \frac{(2x+2y)(x^2+y^2)^2 - 4x(x^2+2xy+by^2)(x^2+y^2)}{(x^2+y^2)^4}$$

Si ha:

$$(2y+2x)(x^2+y^2)^2 - 4y(y^2+2xy+ax)(x^2+y^2) = -(2x+2y)(x^2+y^2)^2 + 4x(x^2+2xy+by^2)(x^2+y^2)$$

$\Rightarrow$

$$(2y+2x)(x^2+y^2) - 4y(y^2+2xy+ax) = -(2x+2y)(x^2+y^2) + 4x(x^2+2xy+by^2)$$

$\Rightarrow$

$$2x^2y + \cancel{2y^3} + \cancel{2x^3} + 2xy^2 - \cancel{4y^3} - 8xy^2 - 4axy = -\cancel{2x^3} - 2xy^2 - 2x^2y - \cancel{2y^3} + \cancel{4x^3} + 8x^2y + 4bxy^2$$

$\Rightarrow$

$$-4x^2y - 4xy^2 - 4bxy^2 - 4axy = 0$$

$\Rightarrow$

$$x^2y + xy^2 + bxy^2 + axy = 0$$

Potrebbe esserci un errore nella traccia

$\Rightarrow$

$$x^2y + xy^2(1+b) + ax^2y = 0$$

$$\Rightarrow x^2y(1+a) + xy^2(1+b) = 0 \quad \Rightarrow \quad \begin{matrix} a=-1 \\ b=-1 \end{matrix}$$

Consideriamo perciò

$$\omega = \frac{y^2+2xy-x^2}{(x^2+y^2)^2}\,dx - \frac{x^2+2xy-y^2}{(x^2+y^2)^2}\,dy$$

Vediamo se ω è esatta:

Vediamo se ω è esatta:

Sia $\gamma = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$

$$\int_\gamma \omega = \int_0^{2\pi} \langle (F_1(\gamma(t)), F_2(\gamma(t))), \gamma'(t) \rangle \, dt =$$

$$= \int_0^{2\pi} (\sin^2(t) + 2\cos(t)\sin(t) - \cos^2(t)) \cdot (-\sin(t)) - (\cos^2(t) + 2\cos(t)\sin(t) - \sin^2(t))(\cos(t)) \, dt =$$

$$= \int_0^{2\pi} -\sin^3(t) \underline{\underline{- 2\cos(t)\sin^2(t)}} + \underline{\cos^2(t)\sin(t)} - \cos^3(t) \underline{- 2\cos^2(t)\sin(t)} + \underline{\underline{\cos(t)\sin^2(t)}} \, dt =$$

$$= \int_0^{2\pi} \underbrace{-\sin^3(t)} \underbrace{- \cos^3(t)} \underbrace{- \cos(t)\sin^2(t)} \underbrace{- \cos^2(t)\sin(t)} \, dt$$

$$\underbrace{-\sin^3(t)} = -\sin \cdot \sin^2 = (1-\cos^2(t)) \Downarrow -\sin(t) + \cos^2(t)\sin(t)$$

$$\underbrace{-\cos^3(t)} = -\cos \cdot \cos^2 = (1-\sin^2(t)) \Downarrow -\cos(t) + \cos(t)\sin^2(t)$$

$$\underbrace{-\cos(t)\sin^2(t)} = \leftarrow \text{Primitiva} \quad -\frac{\sin^3(t)}{3}$$

$$\underbrace{-\cos^2(t)\sin(t)} = \leftarrow \text{Primitiva} \quad \frac{\cos^3(t)}{3}$$

$$\Rightarrow \int_0^{2\pi} -\sin(t) + \cancel{\cos^2(t)\sin(t)} - \cos(t) + \cancel{\cos(t)\sin^2(t)} \cancel{- \cos(t)\sin^2(t)} \cancel{- \cos^2(t)\sin(t)} \, dt$$

$$\Rightarrow \int_0^{2\pi} -\sin(t) - \cos(t) \, dt = \cos(t) - \sin(t) \Big|_0^{2\pi} =$$

$$= (1-0) - (1-0) = 1 - 1 = 0$$

$\Rightarrow$ è esatta

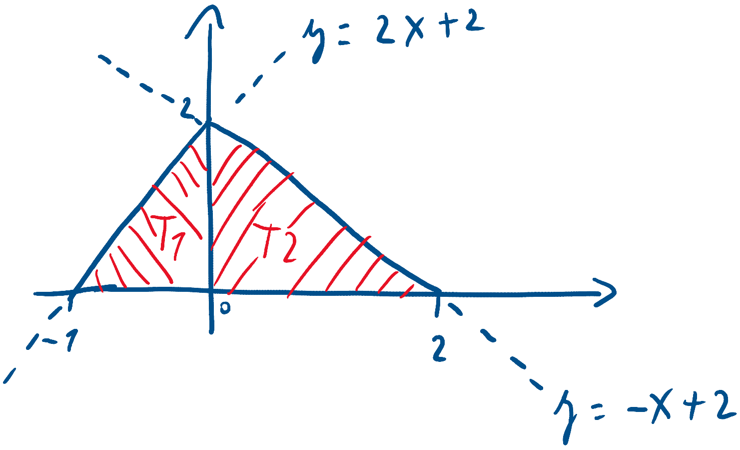
* da finire, resta da calcolare il potenziale *

Esercizio 4 :

Calcolare $\iint_T \frac{y}{1+x}\,dx\,dy$

dove T è il triangolo di vertici $(-1,0)$, $(2,0)$, $(0,2)$

Sol



$$\iint_T \frac{y}{1+x}\,dx\,dy = \iint_{T_1} \frac{y}{1+x}\,dx\,dy + \iint_{T_2} \frac{y}{1+x}\,dx\,dy =$$

$$= \int_{-1}^{0}\int_{0}^{2x+2} \frac{y}{x+1}\,dy\,dx + \int_{0}^{2}\int_{0}^{-x+2} \frac{y}{x+1}\,dy\,dx =$$

$= \dots =$ *da finire*

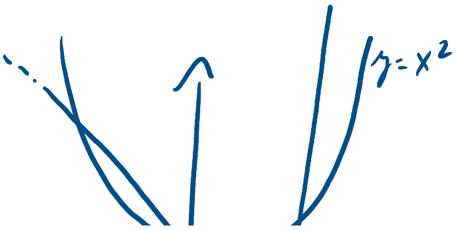
▨

Esercizio 5:

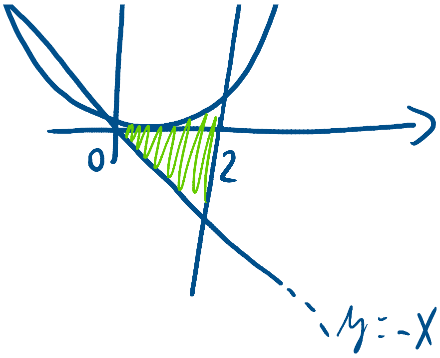
Calcolare la misura di $E = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2,\ -x \leq y \leq x^2\}$

Sol

$m(E) = \iint_E dx\,dy$



$$m(E) = \iint_E dx\,dy$$



$$m(E) = \int_0^2 \int_{-x}^{x^2} dy\, dx =$$

$$= \int_0^2 y \Big|_{-x}^{x^2} dx = \int_0^2 x^2 + x \, dx = \frac{x^3}{3} + \frac{x^2}{2} \Big|_0^2 =$$

$$= \frac{8}{3} + \frac{4}{2} = \frac{8}{3} + 2 = \frac{8+6}{3} = \frac{14}{3}$$

▨

Esercizio 6 :

Calcolare

(a) $\iint_T xy\sqrt{x^2+y^2}\, dx\, dy$

dove T è il triangolo di vertici $(0,0), (1,0), (1,\sqrt{3})$

(b) $\iint_D x^2 + \sin(y)\, dx dy$

dove $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0 , x + y \leq 1\}$

Sol

* Per casa *

▨

4.

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8.

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Esercizio 1:

Sia M il sottoinsieme di $\mathbb{R}^3$ definito dal sistema di equazioni

$$\begin{cases} 2x^2+2y^2-z^2=1 \\ (x-y)^2+z=2 \end{cases}$$

(i) Provare che M è una varietà differenziale 1-dimensionale

(ii) Trovare l'equazione della retta (affine) tangente ad M nel punto $(1,0,1)$

(iii) Trovare i punti di M che sono stazionari per $f(x,y,z)=z$ [ottimizzare f sul vincolo M]

Sol

(i): Siano $g_1(x,y,z) = 2x^2+2y^2-z^2-1=0$

$$g_2(x,y,z) = (x-y)^2+z-2=0$$

Cioè $G=(g_1,g_2)$

$$\Rightarrow JG = \begin{bmatrix} 4x & 4y & -2z \\ 2(x-y) & -2(x-y) & 1 \end{bmatrix}$$

Analizziamo dunque:

(1) $\det \begin{bmatrix} 4x & 4y \end{bmatrix} = -8x(x-y)-8y(x-y)$

$$(1)\quad \det\begin{bmatrix} 4x & 4y \\ 2(x-y) & -2(x-y) \end{bmatrix} = -8x(x-y)-8y(x-y)$$

$$(2)\quad \det\begin{bmatrix} 4x & -2z \\ 2(x-y) & 1 \end{bmatrix} = 4x+4z(x-y)$$

$$(3)\quad \det\begin{bmatrix} 4y & -2z \\ -2(x-y) & 1 \end{bmatrix} = 4y-4z(x-y)$$

Vediamo dove si annullano i tre determinanti:

$$\begin{cases} -8x(x-y)-8y(x-y)=0 \\ 4x+4z(x-y)=0 \\ 4y-4z(x-y)=0 \end{cases} \Rightarrow \begin{cases} -8\big(x(x-y)+y(x-y)\big)=0 \\ 4\big(x+z(x-y)\big)=0 \\ 4\big(y-z(x-y)\big)=0 \end{cases}$$

$$\Rightarrow \begin{cases} \overset{\nearrow y=x}{(x-y)}\overset{\nearrow y=-x}{(x+y)}=0 \\ x+z(x-y)=0 \\ y-z(x-y)=0 \end{cases} \Rightarrow \begin{cases} y=x \\ x=0 \\ y=0 \end{cases} \quad \vee \quad \begin{cases} y=-x \\ x+z(2x)=0 \\ -x-z(2x)=0 \end{cases}$$

$$\Downarrow$$

$$(0,0,z)$$

$$\Downarrow$$

$$\begin{cases} y=-x \\ x+2xz=0 \\ -x-2xz=0 \end{cases} \Rightarrow \begin{cases} y=-x \\ x(1+2z)=0 \\ -x(1+2z)=0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} y=-x \\ x=0 \\ x(1+2z)=0 \end{cases} \quad \vee \quad \begin{cases} y=-x \\ z=-\frac{1}{2} \\ 0=0 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(0,0,z) \qquad\qquad \begin{cases} x=x \\ y=-y \\ z=-\frac{1}{2} \end{cases}$$

$$\Downarrow$$

ü

$(x,-x,-\frac{1}{2})$

Abbiamo trovato i punti $P=(0,0,z)$, $Q=(x,-x,-\frac{1}{2})$

Verifichiamo se P e $Q \in M$:

$$M=\begin{cases} 2x^2+2y^2-z^2=1 \\ (x-y)^2+z=2 \end{cases}$$

$$M(P) \Rightarrow \begin{cases} -z^2=1 \\ z=2 \end{cases} \Rightarrow \text{IMPOSSIBILE} \Rightarrow P \notin M$$

$$M(Q) \Rightarrow \begin{cases} 2x^2+2x^2-\frac{1}{4}=1 \\ 4x^2-\frac{1}{2}=2 \end{cases} \Rightarrow \begin{cases} 4x^2=\frac{5}{4} \Rightarrow x^2=\frac{5}{16} \Rightarrow x=\pm\frac{\sqrt{5}}{4} \\ 4x^2=\frac{5}{2} \Rightarrow x^2=\frac{5}{8} \Rightarrow x=\pm\sqrt{\frac{5}{8}} \end{cases}$$

$\Rightarrow$ IMPOSSIBILE $\Rightarrow Q \notin M$

$\Rightarrow M$ è una varietà

(ii): Vogliamo che

$$J(f(1,0,1)) \cdot \begin{pmatrix} x-1 \\ y \\ z-1 \end{pmatrix} = \vec{0}$$

Sappiamo che $J(f)=\begin{bmatrix} 4x & 4y & -2z \\ 2(x-y) & -2(x-y) & 1 \end{bmatrix}$

$$\Rightarrow J(f(1,0,1))=\begin{bmatrix} 4 & 0 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \; J(G(1,0,1)) \begin{pmatrix} x-1 \\ y \\ z-1 \end{pmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix} =$$

$$= \begin{cases} 4(x-1) - 2(z-1) \\ 2(x-1) - 2y + z - 1 \end{cases}$$

$$\Rightarrow \begin{cases} 4(x-1) - 2(z-1) = 0 \\ 2(x-1) - 2y + z - 1 = 0 \end{cases} \Rightarrow \begin{cases} 2(x-1) - z + 1 = 0 \\ 2x - 2 - 2y + z - 1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x - 2 - z + 1 = 0 \\ 2x - 2y + z = 3 \end{cases} \Rightarrow \begin{cases} 2x - z = 1 \\ 2x - 2y + z = 3 \end{cases} \Rightarrow \begin{cases} 2x = 1 + z \Rightarrow x = \frac{1+z}{2} \\ 1 + z - 2y + z = 3 \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{1+z}{2} \\ 2z - 2y = 2 \end{cases} \Rightarrow \begin{cases} x = \frac{1+z}{2} \\ z - y = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1+z}{2} \\ y = z - 1 \end{cases}$$

(iii): Applichiamo il metodo dei moltiplicatori di Lagrange ad $f(x,y,z) = z$ sul sistema di vincoli $\begin{cases} g_1 = 0 \\ g_2 = 0 \end{cases}$

$$\Rightarrow \begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

$$\nabla f = (0,0,1)$$

$$\nabla g_1 = (4x, 4y, -2z)$$

$$\nabla g_2 = (2(x-y), -2(x-y), 1)$$

$$\Rightarrow \begin{cases} 0 = \lambda_1 4x + \lambda_2 2(x-y) \\ 0 = \lambda_1 4y - \lambda_2 2(x-y) \\ 1 = -\lambda_1 2z + \lambda_2 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 = -\dfrac{2\lambda_2(x-y)}{4} \\ -2\lambda_2(x-y)\cdot y - 2\lambda_2(x-y) = 0 \\ 1 = \lambda_2(x-y)z + \lambda_2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\dfrac{\lambda_2(x-y)}{2} \\ -2\lambda_2(x-y)[y+1] = 0 \\ \lambda_2(z(x-y)+1) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 = -\dfrac{\lambda_2(x-y)}{2} \\ \lambda_2 = \dfrac{1}{z(x-y)+1} \\ -2\dfrac{1}{z(x-y)+1}(x-y)(y+1) = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\dfrac{\lambda_2(x-y)}{2} \\ \lambda_2 = \dfrac{1}{z(x+y)+1} \\ -\dfrac{2(x-y)(y+1)}{z(x-y)+1} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = x \\ \text{IMPOSSIBILE} \end{cases} \vee \begin{cases} y = -1 \\ \lambda_1 = -\dfrac{\lambda_2(x+1)}{2} \\ \lambda_2 = \dfrac{1}{z(x-1)+1} \end{cases} \Rightarrow \begin{cases} y = -1 \\ \lambda_2 = \dots \\ \lambda_1 = -\dfrac{(x+1)}{2z(x-1)+2} \end{cases}$$

???

Esercizio 2:

Si consideri l'ellissoide $E = \left\{ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \right\}$, $a,b,c > 0$

Si consideri l'ellissoide $E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$, $a, b, c > 0$

(i) Mostrare che la funzione $f(x,y,z) = xyz$ ha massimo e minimo assoluti sull'insieme $T = \{(x,y,z) \in E \mid x \geq 0, y \geq 0, z \geq 0\}$

(ii) Servirsi di (i) per calcolare $\sup(g(S))$ e $\inf(g(S))$ dove $S = \{(x,y,z) \in T \mid x > 0, y > 0, z > 0\}$ e $g(x,y,z) = \frac{1}{xyz}$

(iii) Dato $(x,y,z) \in S$ scrivere l'equazione cartesiana del piano tangente in (x,y,z) all'ellissoide E e trovare le intersezioni di tale piano con gli assi coordinati

(iv) Trovare un punto $(x,y,z) \in S$ t.c. il volume del tetraedro racchiuso tra il piano tangente per tale punto all'ellissoide E ed i piani coordinati sia minimo.

<u>Sol</u>

(i): T è compatto

E è compatto, $T \subset E \Rightarrow T$ è limitato

T è chiuso

$\Rightarrow$ Per il Teo di Weierstrass f ammette max e min assoluti su T

Ora $f(x,y,z) = xyz$

$$g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\Rightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} yz = \lambda \frac{2x}{a^2} \\ xz = \lambda \frac{2y}{b^2} \\ xy = \lambda \frac{2z}{c^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{cases}$$

Da finire

9.

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Esercizio 1 :

Sia S la superficie di una sfera di raggio $R>0$

Verificare che:

(i) S è una superficie regolare

(ii) l'area di S vale $4\pi R^2$

Sol

(i): Parametrizziamo in coordinate sferiche

$$\psi : \begin{cases} x = R \sin(\varphi) \cos(\theta) \\ y = R \sin(\varphi) \sin(\theta) \\ z = R \cos(\varphi) \end{cases} \qquad \begin{matrix} \varphi \in [0,\pi] \\ \theta \in [0,2\pi] \end{matrix}$$

Per dimostrare che S è una superficie regolare, bisogna verificare che:

(1) ψ è di classe C^1 [ok]

(2) ψ è iniettiva [ok]

(3) La matrice Jacobiana di ψ ha caratteristica 2 $\forall (\varphi,\theta) \in [0,\pi] \times [0,2\pi]$

$$J\psi(\varphi,\theta) = \begin{bmatrix} x_\varphi & y_\varphi & z_\varphi \\ x_\theta & y_\theta & z_\theta \end{bmatrix} =$$

$$= \begin{bmatrix} R\cos(\varphi)\cos(\theta) & R\cos(\varphi)\sin(\theta) & -R\sin(\varphi) \\ -R\sin(\varphi)\sin(\theta) & R\sin(\varphi)\cos(\theta) & 0 \end{bmatrix}$$

Calcoliamo i determinanti dei minori di ordine 2 :

$$\det \begin{bmatrix} R\cos(\varphi)\cos(\theta) & R\cos(\varphi)\sin(\theta) \\ -R\sin(\varphi)\sin(\theta) & R\sin(\varphi)\cos(\theta) \end{bmatrix} = \dots = R^2 \sin(\varphi)\cos(\varphi)$$

$$\det \begin{bmatrix} R\cos(\varphi)\cos(\theta) & -R\sin(\varphi) \\ -R\sin(\varphi)\sin(\theta) & 0 \end{bmatrix} = \dots = -R^2 \sin^2(\varphi)\sin(\theta)$$

$$\det \begin{bmatrix} R\cos(\varphi)\sin(\theta) & -R\sin(\varphi) \\ R\sin(\varphi)\cos(\theta) & 0 \end{bmatrix} = \dots = -R^2 \sin^2(\varphi)\cos(\theta)$$

Al fine di dimostrare che J_ψ ha rango massimo, calcoliamo la somma dei quadrati dei determinanti:

$$\left(R^2\sin(\varphi)\cos(\varphi)\right)^2 + \left(-R^2\sin^2(\varphi)\sin(\theta)\right)^2 + \left(-R^2\sin^2(\varphi)\cos(\theta)\right) =$$

$$= \dots = R^4 \sin^2(\varphi) \neq 0 \quad \forall \varphi \in (0,\pi)$$

$\Rightarrow$ S è superficie regolare

(ii): $A(S) = \int_S d\sigma = \iint_{[0,\pi]\times[0,2\pi]} \|\psi_\varphi \wedge \psi_\theta\| \, \partial\varphi \, \partial\theta$

$$\Rightarrow \int_0^\pi \int_0^{2\pi} R^2 \sin(\varphi) \, \partial\theta \, \partial\varphi = \int_0^\pi 2\pi R^2 \sin(\varphi) \, \partial\varphi =$$

$$= 2\pi R^2 \left[-\cos(\varphi)\right]\Big|_0^\pi = 2\pi R^2 \cdot 2 = 4\pi R^2$$

▨

Esercizio 2 :

Calcolare l'area del paraboloide $z = \frac{x^2+y^2}{2}$

con $(x,y) \in T = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 8\}$

SOL

Essendo la superficie data da un'equazione cartesiana, una possibile parametrizzazione è:

$$\psi : \begin{cases} x = x \\ y = y \\ z = f(x,y) = \frac{x^2+y^2}{2} \end{cases}$$

$$\Rightarrow \psi_x \wedge \psi_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1)$$

$$\Rightarrow \|\psi_x \wedge \psi_y\| = \sqrt{1 + \|\nabla f\|^2}$$

Per cui

$$A(S) = \int_S d\sigma = \iint_T \sqrt{1+\|\nabla f\|^2} \, dx \, dy$$

Se $f(x,y) = \frac{x^2+y^2}{2} \Rightarrow \begin{pmatrix} f_x = x \\ f_y = y \end{pmatrix}$

$$\Rightarrow A(S) = \iint_T \sqrt{1+x^2+y^2}\;dx\,dy$$

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0, \sqrt{8}] \\ y = \rho\sin(\theta) & \theta \in [0, 2\pi] \end{cases}$$

$$\Rightarrow \int_0^{\sqrt{8}}\int_0^{2\pi} \rho\sqrt{1+\rho^2}\;d\theta\,d\rho = 2\pi\int_0^{\sqrt{8}} \rho\sqrt{1+\rho^2}$$

$$= 2\pi\left[\frac{(1+\rho^2)^{3/2}}{3}\right]\Bigg|_0^{\sqrt{8}} = 2\pi\left(9 - \frac{1}{3}\right) = \frac{52}{3}\pi$$

▨

Esercizio 3:

Calcolare

$\int_S x^2+y^2\,d\sigma$ dove S = porzione di grafico della funzione $g(x,y)=xy$ interna al cilindro di equazione $x^2+y^2=8$

SOL

Parametrizziamo

$$\psi : \begin{cases} x = x \\ y = y \\ z = xy \end{cases}$$

$$\varphi_x = (1, 0, y) \qquad \Rightarrow \qquad \varphi_x \wedge \varphi_y = (-y, -x, 1)$$
$$\varphi_y = (0, 1, x)$$

$$\Rightarrow \| \varphi_x \wedge \varphi_y \| = \sqrt{1 + x^2 + y^2}$$

Per cui

$$\int_S x^2 + y^2 \, d\sigma = \int_T (x^2 + y^2) \cdot \sqrt{1 + x^2 + y^2}$$

Parametrizziamo T in coordinate cilindriche:

$$\varphi : \begin{cases} x = \rho \cos(\theta) \\ y = \rho \sin(\theta) \\ z = z \end{cases} \qquad \begin{array}{l} \theta \in [0, 2\pi] \\ \rho \in [0, \sqrt{8}] \end{array}$$

$$\int_T (x^2 + y^2) \sqrt{1 + x^2 + y^2} = \int_0^{\sqrt{8}} \int_0^{2\pi} \rho^3 \sqrt{1 + \rho^2} \, d\theta \, d\rho =$$

$$= \int_0^{\sqrt{8}} 2\pi \rho^3 \sqrt{1 + \rho^2} \, d\rho =$$

$$= 2\pi \int_0^{\sqrt{8}} \rho^3 \sqrt{1 + \rho^2} \, d\rho = 2\pi \int_0^{\sqrt{8}} \underbrace{\rho^2}_{f(x)} \underbrace{\rho \sqrt{1 + \rho^2}}_{g'(x)} \, d\rho = \quad g'(x) \Rightarrow g(x) = \frac{(1 + \rho^2)^{3/2}}{3}$$

$$= 2\pi \left[\rho^2 \frac{(1 + \rho^2)^{3/2}}{3} - \int_0^{\sqrt{8}} 2\rho \frac{(1 + \rho^2)^{3/2}}{3} \right] =$$

$$= 2\pi \Big[\rho^2 (1 + \rho^2)^{3/2} - 1 \Big(2\rho (1 + \rho^2)^{3/2} \Big] =$$

$$= 2\pi \left[\rho^2 \frac{(1+\rho^2)^{3/2}}{3} - \frac{1}{3} \int 2\rho \left(1+\rho^2\right)^{3/2} \right] =$$

$$= 2\pi \left[\rho^2 \frac{(1+\rho^2)^{3/2}}{3} - \frac{1}{3} \, \frac{2}{5} \left(1+\rho^2\right)^{5/2} \right] \Big|_0^{\sqrt{8}} =$$

$$= 2\pi \left[8 \, \frac{27}{3} - \frac{2}{15} (9)^{5/2} + \frac{2}{15} \right] =$$

$$= 2\pi \left[8 \cdot 9 - \frac{2}{15} (9)^{5/2} + \frac{2}{15} \right] = \frac{949}{15} \pi$$

▨